Random Matrix Approach to Quantum Adiabatic Evolution Algorithms

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We analyze the power of quantum adiabatic evolution algorithms (QAA) for solving random computationally hard optimization problems within a theoretical framework based on the random matrix theory (RMT). We present two types of the driven RMT models. In the first model, the driving Hamiltonian is represented by Brownian motion in the matrix space. We use the Brownian motion model to obtain a description of multiple avoided crossing phenomena. We show that the failure mechanism of the QAA is due to the interaction of the ground state with the "cloud" formed by most of the excited states, confirming that in the driven RMT models, the Landau-Zener scenario of pairwise level repulsions is not relevant for the description of non-adiabatic corrections. We show that the QAA has a finite probability of success in a certain range of parameters, implying the polynomial complexity of the algorithm. The second model corresponds to the standard QAA with the problem Hamiltonian taken from the Gaussian Unitary RMT ensemble (GUE). We show that the level dynamics in this model can be mapped onto the dynamics in the Brownian motion model. However, this driven GUE model always leads to the exponential complexity of the algorithm due to the presence of the long-range multiple time correlations of the eigenvalues. Our results indicate that the weakness of effective transitions is the leading effect that can make the Markovian type QAA successful.

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I. INTRODUCTION

The quantum adiabatic algorithms (QAA) are designed for solving combinatorial search and optimization problems based on the quantum evolution of the relevant systems [1]. These algorithms are based on adiabatic theory of Quantum Mechanics, stating that the quantum state is closely following an instantaneous ground state of a slow-varying in time control Hamiltonian, provided that the initial state was chosen to be a ground state of the initial Hamiltonian. At the initial moment of time the control Hamiltonian has a simple form with the known ground state that is easy to prepare and at the final moment of time it coincides with the "problem" Hamiltonian H_P which ground state encodes the solution of optimization problem in question. For example, in the case of binary optimization problem it can be chosen to reflect the bit-structure and cost spectrum of the problem

$$H_P = \sum_{\mathbf{z}} E_{\mathbf{z}} |\mathbf{z}\rangle\langle\mathbf{z}| \tag{1}$$
$$|\mathbf{z}\rangle = |z_1\rangle_1 \otimes |z_2\rangle_2 \otimes \cdots \otimes |z_n\rangle_n.$$

Here $E_{\mathbf{z}}$ is a cost function defined on a set of 2^n binary strings $\mathbf{z} = \{z_1, \ldots, z_n\}$ $z_j = 0, 1$, each containing n bits. The summation in (1) is over 2^n states $|\mathbf{z}\rangle$ forming the computational basis of a quantum computer with n qubits. State $|z_j\rangle_j$ of the j-th qubit is an eigenstate of the Pauli matrix $\hat{\sigma}_z$ with eigenvalue $1 - 2z_j \pm 1$). If at the end of QAA the quantum state is sufficiently close to the ground state of H_P then the solution to the optimization problem can be retrieved by the measurement.

Different problem arises in simulations of quantum many body systems on quantum computer. In this case quantum Hamiltonian H_P,

$$H_P = \sum_{\mathbf{z},\mathbf{z}'} V_{\mathbf{z}\mathbf{z}'} |\mathbf{z}\rangle\langle\mathbf{z}'|, \tag{2}$$

is not diagonal in the computational basis $\mathbf{z}\rangle$ that is used for binary encoding (examples being quantum spin systems). At the end of adiabatic evolution in QAA the system is brought to the ground state of H_P that is a superposition state $\sum_{\mathbf{z}} c_{\mathbf{z}} |\mathbf{z}\rangle$. Repeating the adiabatic evolution in QAA multiple times and performing a measurement each time at the final moment one can recover the ground state energy as well as the observable properties of the ground state wave function. Similar approach has been recently demonstrated on experiment where the quantum annealing of a disordered magnet was studied [2].

It was shown [5, 25] that the system can be trapped during the QAA in the local minimum of a cost function for a time that grows exponentially in the problem size n. It was also shown [5] that an exponential delay time in quantum adiabatic algorithm can be interpreted as a quantum-mechanical tunneling of an auxiliary large spin system.

The paper [6] demonstrates that tunneling can be avoided by a suitable modification of the QAA evolution "paths" with the same initial and final wave functions. The authors presented a general approach using the "stochastic" paths and numerical results indicated that the HWP may be solved in polynomial time with finite probability. This approach was further developed in [7], where the complete analytic characterization of the "stochastic" path evolution and probability of success was obtained. Moreover, experimental results [2] suggest that quantum annealing can be performed much more efficiently then thermal annealing and quantum tunneling can play a positive role in speeding the convergence process. Similar conclusions were made in the subsequent publication [3] where the qualitative theoretical analysis of the experimental system [2] was developed based on the cascade of Landau-Zener avoided-crossings.

Rigorous analysis of the efficiency of QAA for the solution of hard random optimization problems requires understanding of the ground-state dynamics of the quantum many body Hamiltonian $H(t)$ that is characterized by many complex features, such as quantum phase transitions [4]. On the other hand it is known for many complex many body problems that time-varying $H(t)$ can possess universal properties described within the framework of "driven" random matrix theory (RMT). At present it has not been established to what extent universal features manifest themselves in the lower edge of the spectrum of $H(t)$ for the combinatorial optimization problems such as Satisfiability [1, 4], and others. However, one can ask a different question: assuming certain universal properties of $H(t)$, what effect they will have on the performance of QAA? Solvable models for QAA using RMT methods can provide important insight into the performance of QAA for generic random optimization problems. In this paper we study analytically the dynamics of QAA in the two mutually-related random optimization problems using methods of driven RMT. In Sec. II we introduce relevant physical quantities to describe the complexity of QAA, in Sec. III we introduce the approach to QAA based on driven RMT and method of optimal trajectory, in Sec. IV we introduce the Brownian motion model for $H(t)$, provide motivation for it and describe the dynamics of QAA, Sec. V is a Conclusion.

II. QUANTUM ADIABATIC ALGORITHM

In a standard QAA [1], one specifies the time-dependent *control* Hamiltonian $H(t)$

$$H(t) = \alpha(t)\, H_D + \beta\,(t)\; H_P, \tag{3}$$

with the initial and terminal conditions

$$\begin{aligned} \alpha(0) \;&=\; 1, \qquad \beta\,(0) = 0, \\ \alpha(T) \;&=\; 0, \qquad \beta\,(T) = 1. \end{aligned} \tag{4}$$

The Hamiltonian (3) guides the quantum evolution of the state vector $|\psi(t)\rangle$ from $t = 0$ to $t = T$, the *run time* of the algorithm, and H_P is the "problem" Hamiltonian given in (1). In case when the parametric evolution of the Hamiltonian (3) is sufficiently slow, the Adiabatic theorem assures that the system initially occupying the ground state of the driver $H_D = H\,(0)$, parametrically evolves into the ground state of $H_P = H(T)$. The standard estimate of the run time that provides such adiabatic evolution, is well known and given by [1]

$$T \gtrsim 1/\Delta^2, \tag{5}$$

where Δ is the lowest value of the energy gap between the ground and the first excited adiabatic states.

The "driver" Hamiltonian H_D is designed to cause the transitions between the eigenstates of H_P. According to the above discussion, one prepares the initial state of the system $|\psi(0)\rangle$ to be a ground state of $H(0) = H_D$. It is typically constructed assuming *no* knowledge of the solution of the classical optimization problem and related ground state of H_P. In the simplest case

$$H_D = -C \sum_{j=0}^{n-1} \sigma_x^j, \quad |\psi(0)\rangle = 2^{-n/2} \sum_{\mathbf{z}} |\mathbf{z}\rangle, \tag{6}$$

where σ_x^j is a Pauli matrix for j-th qubit and $C > 0$ is some scaling constant. Consider instantaneous eigenstates $|\phi_k(\tau)\rangle$ of $H(\tau)$ with energies $E_k(\tau)$ arranged in non-decreasing order at any value of $\tau \in (0, 1)$

$$H|\phi_k\rangle = E_k|\phi_k\rangle, \quad k = 0, 1, \ldots, 2^n - 1. \tag{7}$$

Expanding the non-stationary wave function of the system in terms of the instantaneous adiabatic basis (which is complete for any fixed $\tau \in (0,1)$), we obtain ([8]):

$$|\psi(t)\rangle = \sum_{k=0}^{N} C_k(t) \exp\left[-i\int^t dt' E_k(t')\right] |\phi_k(t)\rangle, \tag{8}$$

with $N = 2^n - 1$ complex coefficients $\{C_k(t)\}$ satisfying the normalization condition $\sum_{k=0}^{N} |C_k(t)|^2 = 1$ for $t \in (0,T)$, and we use the units with $\hbar = 1$. Note that each quantity $|C_k(t)|^2$ $(k = 0, ...N)$ is the probability to find the system in the instantaneous adiabatic eigenstate $|\phi_k(t)\rangle$ at time t. According to the above discussion and taking into account the normalization condition, the initial conditions for (8) are given by

$$C_0(0) = 1, \qquad C_k(0) = 0, \quad k = 1, \ldots, N. \tag{9}$$

Under the assumption that the populations of the excited states of the instantaneous Hamiltonian are small, the Schroedinger equation applied to (8) yields [8]

$$\frac{\partial}{\partial t} C_k(t) = \langle \phi_k(t) | \frac{\partial}{\partial t} | \phi_0(t) \rangle \exp\left\{-i\int^t dt' \left[E_k(t') - E_0(t')\right]\right\}, \tag{10}$$

describing the dynamics of the non-adiabatic transitions from the ground state $k = 0$ to the states with $k = 1, \ldots, N$. Taking into account (10), we obtain the probability $P_0(t)$ for the system to remain in adiabatic ground state in the form

$$P_0(t) = 1 - \Gamma(t), \tag{11}$$

with $\Gamma(t)$ being a total probability of escape from the ground state, or failure of the QAA, defined by

$$\Gamma(t) = \sum_{k=1}^{N} \int_0^t dt' \int_0^t dt'' \left(\frac{\partial}{\partial t'}\right)_{k0} \left(\frac{\partial}{\partial t''}\right)_{0k} \exp\left\{-i\int_{t'}^{t''} d\tau \omega_{k0}(\tau)\right\}. \tag{12}$$

In (12), $\left(\frac{\partial}{\partial t}\right)_{k0} \equiv \langle \phi_k(t) | \frac{\partial}{\partial t} | \phi_0(t) \rangle$ and $\omega_{k0}(t) = E_k(t) - E_0(t)$ is the transition frequency between the ground and the k-th excited adiabatic state. In what follows, we will analyze the probability (12) for a particular case of a driven RMT model for the problem Hamiltonian.

III. RANDOM MATRIX ANALYSIS OF QAA

According to the above discussion, the problem Hamiltonian H_P in QAA can have a complex ground state encoding the optimal solution of the initial optimization problem. Since operators H_D and H_P do not commute a total control Hamiltonian $H(t)$ in general can have a complex structure of its eigenvectors and eigenvalues in a certain range of t even when H_P is diagonal in computational basis (1). As a simple "limiting case" of such a complex Hamiltonian, one may study a random Hamiltonian sampled from one of the symmetry ensembles of the Random Matrix Theory (RMT) [9]. In fact it has been shown that the "random" Hamiltonians belonging to a certain symmetry class (say Unitary Ensemble, or GUE), give a good description of the complex physical systems, which do not possess any specific conservation laws and are only characterized by the symmetry class with respect to the change of basis (unitary transformations) [9], [10]. In this section, we shall analyze a specific case of such a "random" control Hamiltonian, when the matrix elements of $H(t)$ evolve according to the Ornstein-Uhlenbeck processes [18] (motivations for such choice of a dynamical model will be provided in a subsequent section)

$$\frac{dH_{kj}(t)}{dt} = -\gamma H_{kj}(t) + \sigma \xi_{kj}, \tag{13}$$

here $\gamma > 0$ describes the dissipation, $\sigma > 0$ the noise amplitude and $z_{kj}(t)$ correspond to N^2 independent Langevin sources

$$\langle \xi_{kj}(t), \xi_{k'j'}(t') \rangle = \delta_{kk'}\delta_{jj'}\delta(t-t'). \tag{14}$$

In (14), δ_{kl} stands for the Kronecker symbol and $\delta(t-t')$ is a Dirac's delta function. The formal solution of the stochastic differential equation (13) is given by

$$H_{kj}(t) = H_{kj}(0)\exp[-\gamma t] + \sigma \int_0^t dt' \xi_{kj}(t') \exp[-\gamma(t-t')], \tag{15}$$

where $H_{kj}(0)$ is an arbitrary Hermitian matrix representing the "initial condition" for the driven random matrix $H_{kj}(t=0) = H_{kj}(0)$.

A. Angular Decomposition in GUE

The "angular coordinates" for the Hermitian matrices H belonging to the General Unitary Ensemble (GUE) [9] are defined by

$$H = U\Lambda U^{+}, \tag{16}$$

where the diagonal matrix $\Lambda_{ij} = \delta_{ij}\lambda_i$ represents the eigenvalues $\{\lambda_i\}$ of H which are invariant with respect to the unitary transformations $U_{ij} = \psi_j^{(i)}$ characterizing the j-th component of the i-th eigenvector belonging to the eigenvalue number i. Therefore, the unitary matrix U represents a particular choice of the basis and the vector Λ is basis-invariant. Note that the unitarity conditions $U^{+}U = UU^{+} = I$ where I is a unit matrix, correspond to the orthonormality and completeness of the basis $\psi^{(i)}$. Differentiating (16) and taking into account the unitarity conditions, we obtain

$$U^{+}dHU = d\Lambda + [d\Omega, \Lambda], \tag{17}$$

where the generalized "angular" motion is defined by

$$d\Omega = U^{+}dU, \tag{18}$$

and [...] is a commutator, i.e. $[d\Omega, d\Lambda] = d\Omega\Lambda - \Lambda d\Omega$. Making use of (17), we obtain the equations of motion for the eigenvalues and angles Ω in the "local" basis with $U = I$, in the form

$$\begin{aligned} \frac{d\lambda_k}{dt} &= \sigma\xi_{kk} - \gamma\lambda_k + \left(\sum_{l\neq k}\frac{\sigma^2}{\lambda_k - \lambda_l}\right), \\ \frac{d\Omega_{kj}}{dt} &= \frac{\sigma}{\lambda_j - \lambda_k}\xi_{kj}. \end{aligned} \tag{19}$$

From (16), it follows that

$$\begin{aligned} \frac{d\Omega_{k0}}{dt} &= \langle\phi_k(t)|\frac{\partial}{\partial t}|\phi_0(t)\rangle = \left(\frac{\partial}{\partial t}\right)_{k0}, \\ \lambda_k(t) - \lambda_0(t) &= E_k(t) - E_0(t) = \omega_{k0}(t), \end{aligned} \tag{20}$$

establishing the connection between the probability (12) and the angular decomposition variables.

B. Optimal Trajectories for Driven RMT

The first of equation (19) describes the many-particle system in $1D$ undergoing the Brownian motion and interacting with the long-range pairwise potential $\sim 1/r$ [9], [10]. Importantly, the dynamics of the eigenvalues $\Lambda(t)$ does not depend on the eigenvector dynamics $\Omega(t)$. Since the velocities of the Brownian particles contain the noise terms, each realization of the general many-particle trajectory in the phase space is drawn from the random distribution characterized by a particular choice of the Langevin source trajectories $\xi_{kj}(t)$. Therefore, the probability of success of the QAA (12) becomes a functional defined on each of the Langevin source trajectories. Since these trajectories are random, this probability P_0 is characterized by the distribution in the general phase space of the problem, with the "distribution density functional" defined on the realizations of the Langevin sources and therefore on the phase coordinates of the system, in the form [23]

$$W\left[\Lambda(t),\Omega(t)\right]=\exp\left\{-S\left[\Lambda(t),\Omega(t)\right]\right\}, \tag{21}$$

with the action functional S defined through the Lagrangian function

$$S=\int_0^T dt' L\left[\Lambda(t'),\dot{\Lambda}(t'),\Omega(t'),\dot{\Omega}(t')\right], \tag{22}$$

where $\dot{\Lambda}=d\Lambda/dt$ and $\dot{\Omega}=d\Omega/dt$, and L is the Lagrangian. Following the standard procedure, the Lagrangian for the system (19) is given by [22]

$$\begin{aligned} L &= \frac{1}{2\sigma^2}\left\{\sum_k\left[\left(\dot{\lambda_k}-v_k\right)^2+\sigma^2 v_k'\right]+\sum_{k,j}\left(\lambda_k-\lambda_j\right)^2\left(\dot{\Omega_{kj}}\right)\right\}^2, \\ v_k &= -\gamma\lambda_k+\left(\sum_{l\neq k}\frac{\sigma^2}{\lambda_k-\lambda_l}\right), \end{aligned} \tag{23}$$

with $v_k'(\Lambda)=\partial v_k(\Lambda)/\partial\lambda_k$. Making use of (12), (19), and (20), we obtain the expected probability of failure $\langle\Gamma(t)\rangle=1-\langle P_0(t)\rangle$ in the form

$$\langle\Gamma(t)\rangle=\frac{\int\int D\Lambda\, D\Omega\, W\left[\Lambda,\Omega\right]\sum_{k=1}^N\int_0^t dt'\int_0^t dt''\dot{\Omega}_{k0}(t')\,\dot{\Omega}_{0k}(t'')\exp\left\{-i\Phi_{k0}\right\}}{\int\int D\Lambda D\Omega\, W\left[\Lambda,\Omega\right]}, \tag{24}$$

where $\Phi_{k0}(t',t'') = \int_{t'}^{t''} d\tau \omega_{k0}(\tau)$ and $\int D\Lambda D\Omega\, W[\Lambda,\Omega]$ denotes the functional integration over all trajectories $\{\Lambda \equiv \Lambda(t),\, \Omega \equiv \Omega(t)\}$ with the weight defined as a functional probability density (21), and the denominator is included into (24) for normalization. From (23), it follows that the "angular" part enters as an additive quadratic term into the Lagrangian, and therefore the integration over the angular variables $\int D\Omega(t)$ in (24) can be done explicitly [23], with the result

$$\frac{\int D\Omega\, W[\Lambda,\Omega]\, \dot{\Omega}_{k0}(t)\, \dot{\Omega}_{0k}(t')}{\int\int D\Lambda D\Omega\, W[\Lambda,\Omega]} = \left(\frac{\sigma^2}{\left[\lambda_k(t) - \lambda_0(t)\right]^2}\right) \delta(t-t'). \tag{25}$$

Combining (24) and (25), we obtain

$$\langle \Gamma(t) \rangle = \sum_{k=1}^{N} \int D\Lambda(t) \frac{\exp\left(-S_{eff}[\Lambda]\right)}{\int D\Lambda' \exp\left(-S_{eff}[\Lambda']\right)} \int_0^t dt' \frac{\sigma^2}{\left[\lambda_k(t') - \lambda_0(t')\right]^2}, \tag{26}$$

where the effective action $S_{eff}[\Lambda]$ is taken with the "truncated" Lagrangian L_{eff}

$$L_{eff} = \frac{1}{2\sigma^2} \sum_k \left[\left(\dot{\lambda_k} - v_k(\Lambda) \right)^2 + \sigma^2 v_k'(\Lambda) \right]. \tag{27}$$

The evaluation of (26) is still very difficult, since it involves the averaging over the unknown level distribution.

For the many-body system with exponential number of particles $N = 2^n$, the distribution of the system's trajectories in the phase space is expected to be sharply peaked about the optimal trajectory, defined from the variation principle [21]. The variational principle takes the form of minimum action

$$S_{\min} = \min_{\{\Lambda(t),\Omega(t)\}} \left\{ S\left[\Lambda(t), \Omega(t)\right] \right\}, \tag{28}$$

Given (23), the optimal trajectories are obtained from the minimum action principle (28) as the solutions of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \left(\frac{\partial L}{\partial q_i}\right), \tag{29}$$

where $q_i = \{\lambda_i\}$ represent the generalized coordinates [11]. Combining (23) and (29), we obtain

$$\ddot{\lambda}_k(t) - \sum_j \dot{\lambda}_j \left(\frac{\partial v_k}{\partial \lambda_j} - \frac{\partial v_j}{\partial \lambda_k} \right) = -\frac{\partial}{\partial \lambda_k} U_{eff}(\Lambda), \tag{30}$$

with the effective potential given by

$$U_{eff}\left(\Lambda\right)=-\frac{1}{2}\sum_{k}\left(v_{k}^{2}\left(\Lambda\right)+\sigma^{2}\frac{\partial v_{k}\left(\Lambda\right)}{\partial\lambda_{k}}\right). \tag{31}$$

Combining (23), (30) and (31) yields

$$\begin{aligned}\ddot{\lambda}_{k}\left(t\right) &= -\frac{\partial}{\partial\lambda_{k}}U_{eff}\left(\Lambda\right),\\ U_{eff}\left(\Lambda\right) &= -\frac{\gamma^{2}}{2}\sum_{k}\lambda_{k}^{2}.\end{aligned} \tag{32}$$

The details of the calculations leading to (32) are presented in *Appendix A*. The effective potential in (32) can also be derived from the Fokker-Plank equation describing the time evolution of the N-particle level distribution function [18]. In general, that procedure leads to the dynamic equations describing the many-particle system in $1D$ with pairwise interactions. In a particular case of the unitary ensemble, the pairwise interaction vanishes and the problem reduces to the N-particle non-interacting system. Therefore, we finally have the following description of the optimal trajectories

$$\ddot{\lambda}_{k}\left(t\right)-\gamma^{2}\lambda_{k}=0. \tag{33}$$

The equations (33) are of the second order and therefore require two initial (or boundary) conditions for each $k=0,...N$. Since (19) are the first order stochastic differential equations, we only have one initial condition for the optimal trajectories

$$\lambda_{k}\left(t=0\right)=\lambda_{k}\left(0\right), \tag{34}$$

for each k and the final points $\lambda_{k}\left(t=T\right)$ are not fixed. We impose the additional boundary conditions at $t=T$ in the form

$$-\gamma\lambda_{k}\left(T\right)+\left(\sum_{l\neq k}\frac{\sigma^{2}}{\lambda_{k}\left(T\right)-\lambda_{l}\left(T\right)}\right)=0, \tag{35}$$

which insure that at $t=T$ the system belongs to the equilibrium unitary ensemble. Specifically, (35) corresponds to the standard GUE distribution [9], [10]. As we show in *Appendix B*, the boundary conditions (35) can also be obtained in the long-time limit for the optimal

trajectories in case when the terminal conditions at $t = T$ are not fixed and have to be self-consistently derived from the optimality property of the trajectories. In fact, we show that the terminal conditions for the "arbitrary" optimal trajectories are exponentially close to the equilibrium described by (35). This implies that the "spread" of the optimal trajectories is exponentially small in the large-time limit, confirming the validity of the approach described above.

The equations (33) with (34) and (35), provide a close set characterizing the optimal trajectories of the initial many-body system. In the subsequent section, the probability of success of the QAA will be evaluated using the solutions for the optimal trajectories.

C. Probability of Success for the QAA

As we have mentioned above, for the system with exponential number of particles N, the distribution of trajectories in the phase space is expected to be "sharp" around the optimal trajectory. Following this argument, we observe that with exponential accuracy [21]

$$\langle \Gamma(t) \rangle = \sum_{k=1}^{N} \int_0^t dt' \frac{\sigma^2}{\left[\lambda_k^*(t') - \lambda_0^*(t')\right]^2}, \tag{36}$$

where $\{\lambda_k^*\}$ correspond to the optimal trajectories. In order to simplify the notations, we will denote $\{\lambda_k^*\} = \{\lambda_k\}$, i.e. drop the superscript. This will not cause the confusion, since we will only consider the optimal trajectories in the rest of the paper. From (33) with (34) and (35), we obtain

$$\lambda_k(t) = \lambda_k(0) \frac{\sinh\left[\gamma(T-t)\right]}{\sinh(\gamma T)} + \lambda_k(T) \frac{\sinh(\gamma t)}{\sinh(\gamma T)}. \tag{37}$$

The final "coordinates" $\{\lambda_k(T)\}$ are defined from (35) and correspond to the standard equilibrium GUE distribution [9], [10] at $t = T$. Substituting (37) into (36), we obtain

$$\langle \Gamma(T) \rangle = \sum_{k=1}^{N} \int_0^T dt' \frac{\sigma^2 \sinh^2(\gamma T)}{\left[\omega_{k0}(0) \sinh\left[\gamma(T-t')\right] + \omega_{k0}(T) \sinh(\gamma t')\right]^2}. \tag{38}$$

The integration in (38) can be done explicitly, with the result

$$\langle \Gamma(T) \rangle = \frac{\sigma^2}{\gamma} \sinh(\gamma T) \sum_{k=1}^{N} \frac{1}{\omega_{k0}(0)\,\omega_{k0}(T)}. \tag{39}$$

Making use of the terminal condition (35), the probability (39) can be estimated as

$$\langle \Gamma (T) \rangle = \sinh (\gamma T) \, \lambda_0 (T) \left\langle \frac{1}{\omega_{k0} (0)} \right\rangle . \tag{40}$$

Since the Hamiltonian at $t = 0$ is given by (6), the average inverse frequency in (40) is bounded

$$\frac{1}{n} \leq \left\langle \frac{1}{\omega_{k0} (0)} \right\rangle \leq 1. \tag{41}$$

On the other hand, according to (35), we have

$$\lambda_0 (T) = \frac{\sigma \sqrt{2N}}{\sqrt{\gamma}}. \tag{42}$$

Since the spectrum of the problem Hamiltonian is bounded in the large$n$ limit, the ground state energy $\lambda_0 (T)$ should not depend on n. This is satisfied if

$$\sigma_0^2 = \frac{\sigma^2}{\gamma} \sim \frac{1}{N}, \tag{43}$$

which is a common scaling in RMT models [9]. As we will see in the subsequent section, the scaling (43) naturally occurs in the QAA framework. In *Appendix C*, we show that the equilibrium N-level distribution only depends on the parameter $\sigma_0^2 = \sigma^2/\gamma$, confirming that the condition (43) is sufficient for the spectrum to be bounded in a finite interval.

Given the estimate (41), the summation over the energy spectrum is performed using the exact equilibrium condition (35), and therefore the result (40) is exact. This means that (40) is based on the exact summation over the discrete spectrum valid for any total number of states N and does not refer to any continuous level distribution arising in the large N limit. In the large N limit, the equilibrium density of states (DOS) for the GUE ensemble is given by the famous "semicircle" law [9], [10]

$$\rho (\lambda) = \frac{2N}{\pi \lambda_0^2} \sqrt{\lambda_0^2 - \lambda^2}, \qquad -\lambda_0 \leq \lambda \leq \lambda_0, \tag{44}$$

with the normalization condition $\int d\lambda \rho (\lambda) = N$. One should note that near vicinity of the ground state, the level distribution functions may have large corrections, which may significantly change DOS near the ground state in comparison to the bulk distribution [16]. However, one can show that the main contribution to the probability (39) comes from the

bulk of the distribution, where there is a large number of states and (44) is valid in the large N limit. Combining (39) and (44), we obtain analogous to (40)

$$\langle \Gamma (T) \rangle = \frac{\sigma^2}{\gamma} \sinh (\gamma T) \left\langle \frac{1}{\omega_{k0} (0)} \right\rangle \frac{2N}{\pi \lambda_0^2} \int\limits_0^{2\lambda_0} \frac{d\omega}{\omega} \sqrt{\omega (2\lambda_0 - \omega)}. \tag{45}$$

From (45), it follows that the integral over the spectrum in the r.h.s. of (45) is accumulated at the upper bound of the integration, since the integrand only has integrable singularity at $\omega = 0$. This indicates that the main contribution to (45) and therefore to (39), comes from the upper part of the spectrum. After evaluation of the integral, (45) yields

$$\langle \Gamma (T) \rangle = \sinh (\gamma T) \left\langle \frac{1}{\omega_{k0} (0)} \right\rangle \frac{2N\sigma^2}{\gamma \lambda_0}, \tag{46}$$

which reduces to (40) after taking (42) into account. Based on the above arguments, we can employ the semicircle distribution to evaluate the average spacing $\left\langle \frac{1}{\omega_{k0}(0)} \right\rangle$. This is done in *Appendix D*, and the probability (40) takes the form

$$\langle \Gamma (T) \rangle = \frac{2}{\pi n} \lambda_0 (T) \sinh (\gamma T) . \tag{47}$$

Since $\gamma T \sim 1$ and the absolute value of the ground state energy $\lambda_0 (T)$ does not depend on n, it follows from (47) that the failure probability of the QAA is $\langle \Gamma (T) \rangle \sim 1/n$, implying that the algorithm can be successful in the large n limit. As we will show in the subsequent section, this is a consequence of the Markovian nature of the Brownian motion model considered above. In the next section, we will discuss this in more details and compare to the results in the non-Markovian case.

The result (47) could not be obtained using a standard estimate (5) with $\Delta = \omega_{10}$ being the smallest gap between the ground state and the first excited level of the total Hamiltonian (3). As it follows from comparison of (39) and (35), the main contribution to the failure probability comes from the bulk of the spectrum, implying that the failure of the QAA occurs due to the interaction of the ground state with the "cloud" formed by all the excited states with $k = 1, ...N$. In other words, it is the large number of possible transitions from the ground state that may cause the failure of the QAA, not the structure of the minimum gap in the energy spectrum. The main reason for this is that the phase factors present in (12) vanish after the integration over the angular variables (25). Note that these phase factors

contain the transition frequencies and lead to the low transition probabilities in adiabatic regime. This is not the case in the Brownian motion model considered above, when all the excited levels work together and the excitation can be effectively viewed as a transition to the "cloud" of the excited states. This is in qualitative agreement with [15], confirming the conclusion that in the driven RMT models, the Landau-Zener mechanism of dissipation [14] is not important. Indeed, the theory of dissipation based on the Landau-Zener mechanism assumes [13] that the transitions happen due to the pairwise "level crossing" events, rather than the "collective" interactions picture that follows from our analysis.

In the regime when the QAA is successful considered above, the average minimal gap is $\Delta \simeq 1/N$, and the formal application of the standard estimate gives $T \gtrsim N^2$, which is has nothing to do with the result (47). One should note that in some cases, the bound (5) can be improved by optimally adjusting the control parameter corresponding to the rate of the quantum adiabatic evolution [1]. In our case, this would correspond to introducing the time dependent rate parameter $\gamma(t)$ in (33), and then optimizing with respect to the function $\gamma(t)$. However, as we show in *Appendix E*, the result (39) is essentially invariant with respect to the choice of $\gamma(t)$, and therefore (47) does not depend on the functional form of the rate parameter. The main reason is that, according to the above discussion, the QAA fails due to the collective interaction of the ground adiabatic state with the cloud of excited states, and therefore there no well defined "dangerous" moments of time corresponding to the avoiding crossing [14], [13].

One may attempt to correct the standard estimate (5) using the average level spacing $\Delta = \langle \omega_{10} \rangle$ in the vicinity of the ground state instead of the average spacing. The point is that the spacing near vicinity of the edges of the spectrum may be significantly enhanced in comparison to the average one [16]. According to the above discussion, these estimates can not be correct, since they assume that the main contribution to the failure probability comes from the low-energy part of the spectrum.

IV. MOTIVATIONS FOR BROWNIAN MOTION MODEL

In this section, we will show that the specification considered in a previous section is actually quite general, since it can be mapped onto (3), with the appropriate choice of the "trajectory" parametrized by the functions $\alpha(t)$ and $\beta(t)$, and the problem Hamiltonian

H_P described by a random matrix belonging to the Gaussian Unitary ensemble (GUE). In general, such mapping yields the time-dependent coefficients $\{\gamma, \sigma\}$ characterizing the Ornstein-Uhlenbeck process. We will specify a particular parametrization that leads the constant coefficients, and therefore the stationary distributions.

Consider the QAA (3) with a random GUE problem Hamiltonian $H_P = V = H(T)$ and the driver $H_D = H(0)$. In what follows, we will refer to this as a "driven GUE" model. Since V belongs to the GUE, its matrix elements are independent and Normally distributed, $V_{kj} \sim N(0, \sigma_0)$. Taking this and (3) into account, we derive the probability distribution for the matrix elements of the Hamiltonian $H(t)$ in the form

$$P[H,t] = C_N(t) \prod_{k,j} \exp\left\{ -\frac{\left[H_{kj} - \alpha(t) H(0)_{kj}\right]^2}{2\sigma_0^2 \beta^2(t)} \right\}, \tag{48}$$

where $C_N(t)$ is the normalization constant. Suppose there is a change of parametrization $\tau = \tau(t)$ in (3) that maps the total Hamiltonian dynamics (3) onto the Ornstein-Uhlenbeck process (13). The process (13) leads to the following probability distribution

$$P[H,\tau] = [4\pi D(\tau)]^{N^2/2} \prod_{k,j} \exp\left\{ -\frac{\left[H_{kj} - \overline{H}_{kj}(\tau)\right]^2}{4D(\tau)} \right\}, \tag{49}$$

with

$$\begin{aligned} \overline{H}_{kj}(\tau) &= H(0)_{kj} \exp(-\gamma\tau), \\ D(\tau) &= \frac{1}{2} \int_0^\tau d\tau' \sigma^2(\tau') \exp\left[-2\gamma(\tau - \tau')\right], \end{aligned} \tag{50}$$

and we assumed that the effective variance $\sigma^2(\tau)$ may be time-dependent. Comparing (48) and (49), we obtain the conditions on the parameters γ and $\sigma^2(\tau)$, in the form

$$\begin{aligned} \exp(-\gamma\tau) &= \alpha(t), \\ \int_0^\tau d\tau' \sigma^2(\tau') \exp(2\gamma\tau') &= \sigma_0^2 \beta^2(t) \exp(2\gamma t), \end{aligned} \tag{51}$$

and therefore

$$\begin{aligned} \tau(t) &= -\frac{1}{\gamma}\ln[\alpha(t)], \\ \sigma^2(\tau) &= \sigma_0^2 \exp(-2\gamma\tau)\frac{d}{d\tau}\left\{\beta^2[t(\tau)]\exp(2\gamma\tau)\right\}, \end{aligned} \tag{52}$$

implying that the effective variance $\sigma^2(\tau)$ is indeed in general time-dependent. In what follows, we consider the QAA (3)

$$H(t) = H(0)\exp(-\gamma t) + V[1-\exp(-\gamma t)], \tag{53}$$

where γ is a real parameter characterizing the speed of the parametric evolution. The parametrization (53) is a particular case of (3) with

$$\alpha(t) = \exp(-\gamma t), \qquad \beta(t) = 1-\exp(-\gamma t), \qquad T \sim \frac{1}{\gamma}, \tag{54}$$

and the parameters $\alpha(t)$ and $\beta(t)$ being the monotonic functions for $t \in [0, +\infty]$ of (53). From (52), we derive the parameters of the corresponding Ornstein-Uhlenbeck process in the form

$$\begin{aligned} \tau(t) &= t, \\ \sigma^2(t) &= 2\gamma\sigma_0^2[1-\exp(-\gamma t)], \end{aligned} \tag{55}$$

implying that the trajectories (53) are mapped onto the Ornstein-Uhlenbeck process with time-dependent variance parameter. From (55), it follows that the time dependence of the effective variance is slow and achieves the "saturation" during the runtime of the algorithm, when $t \sim 1/\gamma$. As we show in *Appendix F*, there is a particular class of trajectories corresponding to the Ornstein-Uhlenbeck process with constant parameters [17].

As it was pointed out in [17], the mapping (52) is exact in a sense that the level distribution obtained from the corresponding Ornstein-Uhlenbeck model is identical to the level distribution of the parametrically driven Hamiltonian (3), at each instant of time. However, the actual level dynamics of the two models are quite different. For this reason, the level

correlations calculated at different time moments are not equivalent for (13) and the corresponding (53). In particular, the driven system (53) contains the "quenched" disorder terms leading to the multiple time correlations, which are absent in a Markovian system (13), [17]. In case of the QAA with the random problem Hamiltonian (53), the multiple time correlators can be easily estimated. Making use of this and the knowledge of the effective level dynamics obtained by mapping onto (13), we will be able to estimate the probability of success for the model (53). The probability of failure for the QAA is obtained from (12) in the form

$$\Gamma(t) = \sum_{k=1}^{N} \int_0^t dt' \int_0^t dt'' \frac{\left[\dot{H}(t')\right]_{k0} \left[\dot{H}(t'')\right]_{0k}}{\omega_{k0}(t')\,\omega_{k0}(t'')} \exp\left\{-i \int_{t'}^{t''} d\tau \omega_{k0}(\tau)\right\}, \tag{56}$$

where we took into account the identity

$$\left(\frac{\partial}{\partial t}\right)_{k0} = \langle \phi_k(t) | \frac{\dot{H}(t)}{\omega_{k0}(t)} | \phi_0(t) \rangle, \tag{57}$$

with $\dot{H}(t) = \frac{dH}{dt}$. In case (53), we have $\frac{dH}{dt} = -\gamma(H - V)$, and (56) yields

$$\begin{aligned} \Gamma(t) &= \gamma^2 \sigma_0^2 \sum_{k=1}^{N} |C_k(t)|^2, \\ C_k(t) &= \int_0^t dt' \frac{\exp\left\{-i \int_0^{t'} d\tau \omega_{k0}(\tau)\right\}}{\omega_{k0}(t')}, \end{aligned} \tag{58}$$

where we took into account the correlations $\langle V_{k0}, V_{0k} \rangle = \sigma_0^2$ and that the Hamiltonian $H(t)$ is diagonal in the adiabatic basis.

The second of (58) represents the transition amplitude, which typically contains two components. In the adiabatic limit, the leading contribution to the integral (58) comes from the endpoints of the integration and corresponds to the corrections to the adiabatic eigenstates due to the finite rate of the parametric transformation (53). The second contribution comes from the actual transitions between the "true" dynamic states with the non-adiabatic corrections to the states taken into account. The "transition" term is exponentially small in the adiabatic limit and exhibits the non-analytical dependence on the adiabaticity parameter ([8]). On the other hand, the "state correction" term typically has a power-law dependence on the adiabaticity parameter.

The general setting of the QAA requires that the solution of the initial optimization problem is encoded in the *adiabatic* ground state of the Hamiltonian [1]. Therefore, in order to evaluate the success of the QAA, one has to take into account both "correction" and "transition" contributions to the integral (58). The correction term gives the leading contribution and therefore determines the success of the QAA. This distinguishes between the approach adopted in the present paper and the one from ([12]). In ([12]), the authors evaluated the failure of QAA based on the results ([13]) for the *dissipation* rate in the driven RMT ensembles. The dissipation was estimated in ([13]) based on the Landau-Zener theory, which takes only transition components in (58) into account. While this was a reasonable approximation to the estimation of the total dissipation rate, this approach is not applicable to the analysis of the QAA. The point is that the evaluation of the QAA performance requires taking into account any deviations from the adiabatic ground state, regardless whether they occur due to the "true" transitions to the "corrected" states, or the corrections to the states themselves. Therefore, it is essential that both contributions to (58) be taken into account. Moreover, from the above discussion, it follows that the "correction" terms give the leading contribution, and therefore the "transition" effects can be neglected in the adiabatic limit. This is consistent with the results of ([15]). The "correction" contribution in (58) is given by ([8])

$$C_k(t) = \frac{-i}{\left[\omega_{k0}(t)\right]^2}\left\{1 - \exp\left[-i\int_0^t dt'\omega_{k0}(t')\right]\right\}. \tag{59}$$

Combining (58) and (59), we obtain the failure probability of QAA

$$\Gamma(T) = 4\gamma^2\sigma_0^2\sum_{k=1}^{N}\frac{1}{\left[\omega_{k0}(T)\right]^4}\sin^2\left[\frac{1}{2}\int_0^T dt'\omega_{k0}(t')\right]. \tag{60}$$

In order to evaluate the failure probability (60), we have to specify the level dynamics of the driven model (53). Making use of the mapping (55) enables us to apply the optimal trajectory methods developed for the Brownian motion model. The optimal trajectories characterized by (37), and we obtain for the transition frequencies along the optimal trajectories in the form

$$\omega_{k0}(t) = \omega_{k0}(0)\frac{\sinh\left[\gamma(T-t)\right]}{\sinh(\gamma T)} + \omega_{k0}(T)\frac{\sinh(\gamma t)}{\sinh(\gamma T)}. \tag{61}$$

Substituting (61) into (60), we obtain

$$\Gamma(T) = 4\gamma^2\sigma_0^2 \sum_{k=1}^{N} \frac{1}{[\omega_{k0}(T)]^4} \sin^2\left[\frac{1}{2}\Phi_{k0}(T)\right], \tag{62}$$

with the total phase $\Phi_{k0}(T)$ defined by

$$\Phi_{k0}(T) = \frac{2}{\gamma}\left[\frac{\omega_{k0}(0) + \omega_{k0}(T)}{2}\right] \coth\left(\frac{\gamma T}{2}\right). \tag{63}$$

Note that the total phase contains the large factor $1/\gamma \sim T$, as expected. According to the estimates from the previous section, the minimal level separations are estimated as $\omega_{k0}(0) \sim 1$ and $\omega_{k0}(T) \sim \frac{1}{N}$. Since in adiabatic limit the rate of parametric evolution is much smaller than the initial transition frequency $\gamma << \omega_{k0}(0)$, the total phase is large and $\sin^2\left[\frac{1}{2}\Phi_{k0}(T)\right] \approx \frac{1}{2}$. Therefore, (62) reduces to

$$\Gamma(T) = 2\gamma^2\sigma_0^2 \sum_{k=1}^{N} \frac{1}{[\omega_{k0}(T)]^4}. \tag{64}$$

From (64), it follows that in order to evaluate the total failure probability of QAA, we need to know the density of states at the end of the evolution process. This is the consequence of the fact that the level separation is minimal at the end of the evolution. The final configuration of the optimal trajectory is given by (35) and corresponds to the standard GUE and the average density of states is given by the semicircle law (44). However, the expression (64) can not be evaluated in a continuous limit analogous to (39), since the corresponding integral diverges on the lower limit. This implies that the sum is accumulated on the lower limit corresponding to the low-frequency transitions, and the structure of DOS near the lower edge of the spectrum may be important. As it was shown in [16], the DOS in GUE near the ground state has the form of the power law

$$\rho(\lambda) \sim \lambda^{q(\lambda)}, \tag{65}$$

where the exponent $q(\lambda) = 1/6$ in the narrow range near the ground state and tends to the limit $q(\lambda) = 1/2$ corresponding to (44), outside that range. Making use of (65), the probability (64) can be estimated as

$$\Gamma(T) = 2\gamma^2\sigma_0^2 \sum_{k=1}^{N} \frac{1}{k^4 [\Delta\omega_k]^4}, \tag{66}$$

with

$$\Delta\omega_k = (1/N)\left[\rho\left(\omega\right)\right]^{-1} \sim (1/N)\left[\Delta\omega_k\right]^{-q} k^{-q}. \tag{67}$$

Combining (66) and (67), we obtain

$$\Gamma\left(T\right) = 2\gamma^2\sigma_0^2\ N^p \sum_{k=1}^{N} \frac{1}{k^p} \approx \frac{2}{p-1}\gamma^2\sigma_0^2\ N^p, \tag{68}$$

where $p = 4/\left(1+q\right) > 2$. The estimate of the power of QAA is defined from the requirement that the failure probability (68) is small. Taking into account (54) and that in the GUE $\sigma_0^2 \sim 1/N$, we estimate the runtime of the QAA

$$T >> N^a, \qquad a = \frac{3-q}{2\left(1+q\right)}, \tag{69}$$

where the exponent q comes from (65). From (69), it follows that $17/14 \leq a \leq 5/6$, where the upper and lower limits correspond to $q = 1/6$ and $q = 1/2$, respectively. The estimate (69) implies the exponential complexity of the QAA with $a \simeq 1$.

Comparing (47) and (68), we observe that the Markovian evolution model (13) may lead to polynomial complexity, whereas the driven GUE model (53) always leads to the exponential complexity of QAA. This has a simple intuition. According to the above discussion, the main difference between the models (13) and (53) comes from different multiple time level correlations. In the Markovian model (13), there are only short-range correlations, whereas in the driven RMT model (53), the long-range multiple time correlations are present. On one hand, this leads to the complete cancellation of phases, which removes the exponentially small factors for the transitions to the highly excited states and leads to the diffusion type of dissipation process in (13). On the other hand, the absence of the long-range multiple time correlations leads to much weaker effective interaction between the states in the Markovian model (13) as opposed to (53). The comparison of (47) and (68) indicates that the "weakness" of effective transitions is the leading effect that can make the Markovian type QAA (13) successful.

One should note that the driven GUE model may be obtained from the QAA (3) with generally non-random problem Hamiltonian H_P under certain conditions imposed on the problem Hamiltonian. Namely, the assumption is that H_P is presented by a "full matrix" in the adiabatic basis and does not possess any specific structure, implying the lack of any additional integrals of motion [19], [20], [17]. This leads to the possibility of averaging over

the "quenched disorder" introduced by the static fluctuations of the matrix elements, which makes the problem equivalent to the driven GUE model considered above [17]. Essentially, the assumption implies the presence of multiple time correlators with large correlation time, and basis invariance typical for the static RMT ensembles.

In order to establish the connection with the specific combinatorial problems, note that the control Hamiltonian $H(t)$ in QAA (3) may be quite far from the basis-invariant GUE. From the above analysis, it follows that the two identical level distributions may correspond to completely different QAA performance. This difference can only be captured considering the multiple time correlations of the order higher than the second one [17]. In particular, the above results indicate that the applicability of the static RMT should be examined making use of the multiple time correlations. In particular, it can not be uniquely verified based on the Brody parameter defined through the single-time nearest neighbor level distributions [12].

V. CONCLUSION

We analyze the power of quantum adiabatic algorithms (QAA) for solving computationally hard optimization problems within a theoretical framework based on the random matrix theory (RMT). We present two types of the driven RMT models.

In the first model, the "driving" Hamiltonian is represented by Brownian motion in the matrix space. We use the Brownian motion model to obtain a description of multiple avoided crossing phenomena. The model allows for the close-form analytical treatment in adiabatic approximation, within the method of optimal trajectories [23]. We show that the failure mechanism of the QAA is due to the interaction of the ground state with the "cloud" formed by all the excited states. This confirms that in the driven RMT models, the Landau-Zener mechanism of dissipation based on the assumption of pairwise level interactions, is not important (cf. [15]). We show that the QAA may have a finite probability of success in a certain range of parameters, implying the possibility of *polynomial* complexity of the algorithm. This model can be viewed as a relatively "mild" test on the general QAA performance for the computationally hard optimization problems. In this case, the quantum evolution is not affected by the "accidental" level crossing phenomenon that may lead to the QAA failure in standard models [13], and the only reason for the possible QAA failure is a

complex structure of the control Hamiltonian. We also note that the number of independent parameters in the control Hamiltonian is exponentially large and that formally creates a challenge for its experimental implementation. However, as mentioned above, this Hamiltonian can reflect certain universality properties of realistic physical many body Hamiltonians. This question is referred to future studies.

The second model corresponds to the standard QAA with the problem Hamiltonian taken from the Gaussian Unitary RMT ensemble (GUE). We show that the eigenvalues and eigenvector dynamics in this driven GUE model can be mapped onto the dynamics in the Brownian motion model considered before. This enables us to apply the optimal trajectories approach developed for the previous case, to the driven RMT model. However, the driven RMT and the Markovian Brownian motion models have different structure of the intertemporal level correlations, and this leads to different performance of the QAA. The Brownian motion model may lead to polynomial complexity, whereas the driven GUE model always gives the exponential complexity of the algorithm. In the Markovian Brownian motion model, there are only short-range correlations, whereas in the driven RMT model, the long-range multiple time (intertemporal) correlations are present. The absence of the long-range intertemporal correlations leads to much weaker effective interaction between the states in the Markovian model (13) as opposed to (53). Our results indicate that this "weakness" of effective transitions is the leading effect that may in principle make the Markovian type QAA (13) successful. Regarding the connection with the specific combinatorial problems, the above analysis suggests that the applicability of the static RMT should be studied using the multiple time correlations. In particular, it can not be established based on the Brody parameter defined through the single-time nearest neighbor level distributions.

Finally, we note that even when control Hamiltonian does not posses basis-invariant properties explored in this paper we expect that one can use a method of optimal trajectory for random time-dependent Hamiltonians presented above which is based on the angular decomposition described in Sec. IIIA.

VI. APPENDIX A

According to (31) , the effective potential is given by

$$U_{eff}\left(\Lambda\right)=-\frac{1}{2}\sum_{k}\left(v_{k}^{2}\left(\Lambda\right)+\sigma^{2}\frac{\partial v_{k}\left(\Lambda\right)}{\partial\lambda_{k}}\right), \tag{70}$$

with the velocities $v_k\left(\Lambda\right)$ defined by (23). From (23) and (70), we obtain

$$\begin{aligned} U_{eff}\left(\Lambda\right) \;=\; & -\frac{\gamma^{2}}{2}\sum_{k}\lambda_{k}^{2}-\gamma\lambda_{k}\sum_{k,l\neq k}\frac{\sigma^{2}}{\lambda_{k}-\lambda_{l}}- \\ & -\frac{1}{2}\left[\sum_{k,l\neq k,l'\neq k}\frac{\sigma^{4}}{\left(\lambda_{k}-\lambda_{l}\right)\left(\lambda_{k}-\lambda_{l'}\right)}-\sum_{k,l\neq k}\frac{\sigma^{4}}{\left(\lambda_{k}-\lambda_{l}\right)^{2}}\right]. \end{aligned} \tag{71}$$

Under the cyclic permutation of the indices $\{k,l.l'\}$, we have

$$\frac{1}{\left(\lambda_{k}-\lambda_{l}\right)\left(\lambda_{k}-\lambda_{l'}\right)}+\frac{1}{\left(\lambda_{l}-\lambda_{l'}\right)\left(\lambda_{l}-\lambda_{k}\right)}+\frac{1}{\left(\lambda_{l'}-\lambda_{k}\right)\left(\lambda_{l'}-\lambda_{l}\right)}=0,$$

therefore the first term in the square brackets in the r.h.s. of (71) reduces to

$$\sum_{k,l\neq k,l'\neq k}\frac{\sigma^{4}}{\left(\lambda_{k}-\lambda_{l}\right)\left(\lambda_{k}-\lambda_{l'}\right)}=\sum_{k,l\neq k}\frac{\sigma^{4}}{\left(\lambda_{k}-\lambda_{l}\right)^{2}}, \tag{72}$$

and the two terms in the square brackets in (71) sum up to zero. The second term in the r.h.s. of (71) is given by

$$\gamma\sigma^{2}\sum_{k,l\neq k}\frac{\lambda_{k}}{\lambda_{k}-\lambda_{l}}=\gamma\sigma^{2}\frac{N\left(N-1\right)}{2}=const. \tag{73}$$

Combining (71), (72) and (73) finally yields

$$U_{eff}\left(\Lambda\right)=-\frac{\gamma^{2}}{2}\sum_{k}\lambda_{k}^{2},$$

identical to (32) from the text.

VII. APPENDIX B

According to (33), the optimal trajectories satisfy

$$\ddot{\lambda}_k(t) - \gamma^2 \lambda_k = 0. \tag{74}$$

The equations (33) are of the second order and therefore require two initial (or boundary) conditions for each $k = 0, ...N$. Suppose that we only have one initial condition for the optimal trajectories

$$\lambda_k(t=0) = \lambda_k(0), \tag{75}$$

for each k and the final points $\lambda_k(t=T)$ are not fixed. Following the standard procedure [11], we impose the additional "transversality" boundary conditions at the final points

$$\left(\frac{\partial L}{\partial \dot{q}_i}\right)_{t=T} = 0, \tag{76}$$

which yield the boundary conditions for our problem

$$\dot{\lambda}_k(T) = v_k[\Lambda(T)] = -\gamma \lambda_k(T) + \left(\sum_{l \neq k} \frac{\sigma^2}{\lambda_k(T) - \lambda_l(T)}\right). \tag{77}$$

The equations (33) with (75) and (77), provide a close set. From (33) with (75) and (77), we obtain

$$\lambda_k(t) = \lambda_k(0) \frac{\sinh[\gamma(T-t)]}{\sinh(\gamma T)} + \lambda_k(T) \frac{\sinh(\gamma t)}{\sinh(\gamma T)}. \tag{78}$$

The final "coordinates" $\{\lambda_k(T)\}$ are defined from (33) and (77) as solutions of the self-consistent equation

$$\lambda_k(T) = \lambda_k(0) \exp(-\gamma T) + \frac{[1 - \exp(-2\gamma T)]}{\gamma} \left(\sum_{j \neq k} \frac{\sigma^2}{\lambda_k(T) - \lambda_j(T)}\right), \tag{79}$$

which indicates that in the large-time limit $\gamma T >> 1$, the eigenvalues $\{\lambda_k(T)\}$ are separated, i.e. the degeneracy is completely lifted, even though the initial distribution $\{\lambda_k(0)\}$ may have a significant degree of degeneracy. Specifically, (79) leads to the standard equilibrium GUE distribution described by (35) in the large-time limit.

VIII. APPENDIX C

In this Appendix, we will derive the equilibrium N-particle distribution for the levels and show that it is indeed an equilibrium GUE distribution for any choice of the "rate" parameter $\alpha(t)$ introduced in the text. According to the first of equation (19), the non-stationary N-level distribution function $P(\Lambda, t)$ satisfies the Fokker-Plank equation [18]

$$\frac{\partial P(\Lambda,t)}{\partial t}+\frac{\partial}{\partial\lambda_k}\left[v_k(\Lambda)P(\Lambda,t)-\frac{\sigma^2}{2}\frac{\partial P(\Lambda,t)}{\partial\lambda_k}\right], \tag{80}$$

and the equilibrium distribution is given by

$$P_{eq}(\Lambda)=C\exp\left[-\frac{2}{\sigma^2}V_{eff}(\Lambda)\right], \tag{81}$$

with a normalization constant C and the effective potential

$$V_{eq}(\Lambda)=-\int^{\Lambda}d\lambda_k v_k(\Lambda)=\frac{\gamma}{2}\sum_k\lambda_k^2-\sigma^2\sum_{k,l\neq k}\log|\lambda_k-\lambda_l|. \tag{82}$$

Combining (81) and (82), we finally obtain

$$P_{eq}(\Lambda)=C\exp\left[\sum_{k,l\neq k}\log(\lambda_k-\lambda_l)^2-\frac{\gamma}{\sigma^2}\sum_k\lambda_k^2\right], \tag{83}$$

corresponding to the standard GUE distribution [9] for any choice of parameters γ and σ.

IX. APPENDIX D

In this Appendix, we employ the semicircle distribution to evaluate the probability (39). According to (6), the levels of $H(0)=H_D$ are equally spaced with $\omega_{k0}(0)=k$ and the degeneracy of the k-th level is

$$g_k=\binom{n}{k}=\frac{n!}{(n-k)!k!}. \tag{84}$$

In the continuous limit of (44), we have

$$\omega_{k0}(0)=k(\omega)=\frac{n}{\lambda_0}\omega, \tag{85}$$

where the last equality follows from the observation that the width of the spectrum at $t = 0$ is $2\omega_{n0}(0) = 2n$ and there are $2n + 1$ distinct levels, whereas the spectrum at $t = T$ is bounded within the range of $2\lambda_0$. Combining (39) and (44), we obtain

$$\langle \Gamma(T) \rangle = \frac{\sigma^2}{\gamma} \sinh(\gamma T) \frac{2N}{\pi \lambda_0^2} \int_0^{2\lambda_0} \frac{d\omega}{\omega} \sqrt{\omega(2\lambda_0 - \omega)} \frac{g_{k(\omega)}}{k(\omega)}, \tag{86}$$

with the degeneracy $g_{k(\omega)}$ given by (84) and (85). In the large n limit, the degeneracy (84) has a sharp peak at $k = n/2$, and (86) reduces to

$$\langle \Gamma(T) \rangle = \frac{\sigma^2}{\gamma} \sinh(\gamma T) \frac{2N}{\pi \lambda_0^2} \int_0^{2\lambda_0} \frac{d\omega}{\omega} \sqrt{\omega(2\lambda_0 - \omega)} \frac{\lambda_0}{n} \frac{g(\omega)}{\omega}, \tag{87}$$

with

$$g(\omega) = \sqrt{\frac{n}{2\pi}} \exp\left[-\frac{n(\omega - \lambda_0)^2}{2\lambda_0^2} \right]. \tag{88}$$

Evaluating (87) in the large n limit, we obtain

$$\langle \Gamma(T) \rangle = \frac{2}{\pi n} \frac{\sigma^2}{\gamma} \sinh(\gamma T) \frac{2N}{\lambda_0}, \tag{89}$$

which reduces to (47) after taking (42) into account.

X. APPENDIX E

In this Appendix, we show that the estimate (47) for the power of the QAA algorithm given in the text for a particular case of the constant parameter characterizing the rate of changing the Hamiltonian $\alpha \equiv const$, can not be improved by introducing the time-dependent rate $\gamma(t)$.

The failure probability of the QAA is given by

$$\langle \Gamma(t) \rangle = \sum_{k=1}^{N} \int_0^t dt' \frac{\sigma^2}{[\omega_{k0}(t')]^2}, \tag{90}$$

where $\omega_{k0}(t) = \lambda_k(t) - \lambda_0(t)$ are the transition frequencies defined corresponding to the optimal trajectories $\{\lambda_k(t)\}$.

$$\ddot{\lambda}_k(t) - \gamma^2(t) \lambda_k = 0. \tag{91}$$

Analogous to (37), the solution of (91) can be presented in the form

$$\lambda_k(t) = \lambda_k(0)\,\varphi_1(t) + \lambda_k(T)\,\varphi_2(t), \tag{92}$$

where $\{\varphi_1(t), \varphi_2(t)\}$ are the two linearly-independent solutions of the second-order ODE (91) satisfying the following boundary conditions

$$\begin{aligned} \varphi_1(0) &= 1; \qquad \varphi_1(T) = 1; \\ \varphi_2(0) &= 0; \qquad \varphi_2(T) = 1. \end{aligned} \tag{93}$$

In this case, it is well known [24] that for the system (91), the Wronskian $W(\gamma)$ defined by

$$W(\gamma) = \varphi_1(t)\frac{d\varphi_2(t)}{dt} - \varphi_2(t)\frac{d\varphi_1(t)}{dt} = [\varphi_2(t)]^2\frac{d}{dt}\left[\frac{\varphi_1(t)}{\varphi_2(t)}\right], \tag{94}$$

is an integral of motion. Substituting (92) into (90), we obtain analogous to (38)

$$\langle\Gamma(T)\rangle = \sum_{k=1}^{N}\int_0^T dt\frac{\sigma^2}{[\omega_{k0}(0)\,\varphi_1(t) + \omega_{k0}(T)\,\varphi_2(t)]^2}. \tag{95}$$

Making use of (94), (95) reduces to

$$\langle\Gamma(T)\rangle = \sum_{k=1}^{N}\int_0^{+\infty}\frac{du}{W(\gamma)}\frac{\sigma^2}{[\omega_{k0}(0)\,u + \omega_{k0}(T)]^2}, \tag{96}$$

where we made the substitution $u = \varphi_1(t)/\varphi_2(t)$ and took (93) into account. After the integration (96) yields

$$\langle\Gamma(T)\rangle = \frac{\sigma^2}{W(\gamma)}\sum_{k=1}^{N}\frac{1}{\omega_{k0}(0)\,\omega_{k0}(T)}, \tag{97}$$

leading to the same bound on the runtime as the estimate (47). Note that for the case of constant $\gamma = 1/T$ considered in the text, the basis functions $\{\varphi_1(t), \varphi_2(t)\}$ lead to (37), and the Wronskian is given by $W(\gamma) = \gamma/\sinh(\gamma)$. In this case, (97) reduces to (39) from the text.

XI. APPENDIX F

Following [17], consider the QAA (3)

$$H(t) = \cos(\Omega t)\, H_0 + \sin(\Omega t)\; V, \tag{98}$$

where Ω is a real parameter characterizing the "speed" of the parametric evolution. The parametrization (98) is a particular case of (3) with

$$\alpha(t) = \cos(\Omega t), \qquad \beta(t) = \sin(\Omega t), \qquad T = \frac{\pi}{2\Omega}, \tag{99}$$

and the parameters $\alpha(t)$ and $\beta(t)$ being the monotonic functions of time during the runtime $t \in [0, T]$ of (98). From (52), we derive the parameters of the corresponding Ornstein-Uhlenbeck process in the form

$$\begin{aligned} \tau(t) &= -\frac{1}{\gamma} \ln[\cos(\Omega t)], \\ \sigma^2 &= 2\gamma\sigma_0^2, \end{aligned} \tag{100}$$

implying that the trajectories (98) are mapped onto the Ornstein-Uhlenbeck process with constant parameters [17].

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